M252 EXAM 2005 — SOLUTIONS

1) a) i) For a pdf we require $f(x) \geq 0$ everywhere, & $\int_{\mathbb{R}} f(x)\,dx = 1$.

Here, $1-x^2 > 0$ for $-1 < x < 1$, so if c exists it must be positive. Now

$$\int_{\mathbb{R}} f(x)\,dx = c\int_{-1}^{1}(1-x^2)\,dx = 4c/3$$

$\Rightarrow$ $c = 3/4$ yields a pdf.

[Di]stribution function is $F(x) = \int_{-\infty}^{x} f(u)\,du$

$$= \begin{cases} 0 & x \leq -1 \\ (3x - x^3 + 2)/4 & -1 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

ii) Since $0.5 - x^2$ has roots in the range $(-1,1)$, only $c=0$ can yield $f(x) \geq 0$ everywhere, & in this case $\int f(x)\,dx = 0$ $\Rightarrow$ no value of c yields a pdf.

iii) $\int_{\mathbb{R}} f(x)\,dx = c\int_{-1}^{1}(1-x^2)^{-1}\,dx$, which is infinite unless $c=0$. Again, no value of c yields a pdf.

b) Since f is a pdf, we have $\int_0^1 (a+bx^2)\,dx = 1 \Rightarrow a + \frac{b}{3} = 1$
Also, $E(X) = 3/5 \Rightarrow \int_0^1 x(a+bx^2)\,dx = 3/5 \Rightarrow \frac{a}{2} + \frac{b}{4} = 3/5$

Solving these simultaneous equations yields $a = 3/5,\ b = 6/5$

2) a) X takes values $0, 1, 2, \ldots$

$$P(X=k) = P(k \text{ successful pulls, then a failure})$$
$$= \underline{\underline{(1-p)^k p \qquad (k = 0, 1, 2, \ldots)}}$$

b) The light will be off when the switch jams if X is even. We have

$$P(X \text{ is even}) = \sum_{r=0}^{\infty} P(X=2r) = p \sum_{r=0}^{\infty} (1-p)^{2r}$$
$$= p/[1-(1-p)^2]$$
$$= p/[2p-p^2] = \underline{\underline{1/2-p \text{ as required}}}$$

c) PGF is $\Pi(s) = E(s^X) = \sum_{k=0}^{\infty} s^k P(X=k)$

$$= p \sum_{k=0}^{\infty} [s(1-p)]^k = \underline{\underline{p[1-s(1-p)]^{-1}}}$$

$E(X) = \Pi'(1)$, and $\Pi'(s) = p(1-p)[1-s(1-p)]^{-2}$

So $E(X) = p(1-p)/p^2 = \underline{\underline{(1-p)/p}}$

d) Distribution function is

$$F(k) = P(X \leq k) = 1 - P(X > k) = 1 - P(X \geq k+1)$$
$$= 1 - P(\text{first } k+1 \text{ pulls are successful})$$
$$= \underline{\underline{1-(1-p)^{k+1} \quad (k \geq 0)}}$$

3) a) i) Let X be the lifetime. We require

$$P(X>40) = P\left(\frac{X-34}{4} > \frac{40-34}{4}\right) = P(Z>1.5) \text{ where } Z \sim N$$
$$= 1 - 0.9332 = \underline{\underline{0.0668}} \text{ (tables)}$$

ii) $$P(30 < X \leq 35) = P(-1 < Z \leq 0.25)$$
$$= P(Z \leq 0.25) - P(Z \leq -1)$$
$$= 0.5987 - (1 - P(Z \leq 1))$$
$$= 0.5987 - 0.1587 = \underline{\underline{0.44}}$$

iii) $$P(X>40 \mid X>30) = P(X>40 \cap X>30) / P(X>30)$$
$$= P(X>40) / P(X>30) = 0.0668 / 0.8413$$
$$= \underline{\underline{0.0794}}$$

b) i) Let Y be the # of tyres failing within the first x kilometres; then $Y \sim Bin(4, p)$, with $p = P(X \leq x) = \Phi\left(\frac{x-34}{4}\right)$. Hence

$$P(Y=0) = (1-p)^4 = \underline{\underline{\left[1 - \Phi\left(\frac{x-34}{4}\right)\right]^4}}$$

ii) Let T be the time until the first tyre failure. The distribution function of T is

$$F(t) = P(T \leq t) = P(\text{at least 1 failure by time } t)$$
$$= 1 - \left[1 - \Phi\left(\frac{t-34}{4}\right)\right]^4 \text{ from previous result.}$$

Differentiating, the density of T is

$$f(t) = \frac{dF}{dt} = \underline{\underline{\left[1 - \Phi\left(\frac{t-34}{4}\right)\right]^3 \phi\left(\frac{t-34}{4}\right)}}$$

4) a) i) This is a gamma distribution with shape parameter ν and scale parameter η.

The mean is ν/η, & the variance is ν/η^2.

ii) $M(t) = E[e^{tX}] = \frac{\eta^\nu}{\Gamma(\nu)} \int_0^\infty x^{\nu-1} e^{-(\eta-t)x}\,dx$

$= \quad (u = (\eta - t)x, \; dx = du/(\eta-t))$

$\frac{\eta^\nu}{\Gamma(\nu)} \int_0^\infty \left(\frac{u}{\eta-t}\right)^{\nu-1} e^{-u} \cdot \frac{du}{\eta-t}$

$= \left(\frac{\eta}{\eta-t}\right)^\nu \cdot \frac{1}{\Gamma(\nu)} \underbrace{\int_0^\infty u^{\nu-1} e^{-u}\,du}_{\Gamma(\nu)}$

$= (1 - t/\eta)^{-\nu}$ as required

$E(X) = M'(0)$ & $E(X^2) = M''(0)$. Now
$M'(t) = \nu/\eta\,(1-t/\eta)^{-\nu-1}$, & $M''(t) = \frac{\nu(\nu+1)}{\eta^2}(1-t/\eta)^{-\nu-2}$.

So $E(X) = \nu/\eta$ (as before)
and $E(X^2) = \nu(\nu+1)/\eta^2$
$\Rightarrow Var(X) = \frac{\nu(\nu+1)}{\eta^2} - \frac{\nu^2}{\eta^2} = \nu/\eta^2$ (as before)

b) $M_S(t) = \prod_{i=1}^n M_{X_i}(t)$ using the given result

$= \prod_{i=1}^n \left(1 - \frac{t}{\eta_i}\right)^{-\nu_i}$

If $\eta_1 = \cdots = \eta_n = \eta$, we have $M_S(t) = \prod_{i=1}^n \left(1 - \frac{t}{\eta}\right)^{-\nu_i}$

$= (1 - t/\eta)^{-\sum_{i=1}^n \nu_i}$, which is the MGF of a gamma distribution with parameters $\sum_{i=1}^n \nu_i$ and η.

5) a) $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $s^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right]$

	No nitrogen	Nitrogen
$\sum_{i=1}^{n} x_i$	3.99	5.45
$\sum_{i=1}^{n} x_i^2$	1.6397	3.1161
$\bar{x}$	0.399g	0.545g
s	0.0728g	0.1273g

b) Under $H_0: \sigma_0^2/\sigma_1^2 = 1$, the test statistic $F = s_0^2/s_1^2$ is distributed as $F_{9,9}$. The upper $2\frac{1}{2}\%$ point of this distribution is approx $\frac{1}{2}(4.102 + 3.964)$ (linear interpolation from Table 12c) $= 4.033$. The lower $2\frac{1}{2}\%$ point is $1/4.033 = 0.2480$. So we will accept H_0 if $0.2480 \le s_0^2/s_1^2 \le 4.033$; otherwise reject.

The observed value of F is 0.327, which is in the acceptance region $\Rightarrow$ accept H_0 & conclude that the data are consistent with equal variances in the underlying distributions

c) 95% CI is $\bar{x}_1 - \bar{x}_0 \pm \left(t \times s_p\sqrt{\frac{1}{n_0} + \frac{1}{n_1}}\right)$, where $t = 2.101$ is the upper 2.5% point of a t_{18} distribution, and $s_p^2 = \left[(n_0-1)s_0^2 + (n_1-1)s_1^2\right]/(n_0+n_1-2)$
$= 0.01075$

$\Rightarrow$ CI is $0.146 \pm (2.101 \times 0.1037/\sqrt{5})$

$= (0.049, 0.243)$

Since the CI excludes zero, there is evidence that nitrogen affects the growth rate of seedlings

6) a) $MSE_\theta(T) = E_\theta\left[(T-\theta)^2\right]$

Let $\mu = E_\theta(T)$; then the bias of T is $b_\theta(T) = \mu - \theta$.
Now

$$MSE_\theta(T) = E_\theta\left[(T-\theta)^2\right] = E_\theta\left[(T-\mu+\mu-\theta)^2\right]$$

$$= E_\theta\left[(T-\mu)^2\right] + 2(\mu-\theta)\underbrace{E_\theta\left[T-\mu\right]}_{=0} + (\mu-\theta)^2$$

$$= Var_\theta(T) + b_\theta^2(T), \text{ as required}$$

b) i) $X \sim Bin(n,p)$, so $E(X) = np$ and $E(T^{-1}) = E(X/n)$ $= p$, so T^{-1} is unbiased for p, as required

$Var(T^{-1}) = Var(X/n) = \frac{1}{n^2}Var(X) = p(1-p)/n$
So $MSE_p(T^{-1}) = p(1-p)/n$

ii) Since T^{-1} takes values $0, 1/n, \ldots, n/n$, T itself takes values $\infty, n, n/2, \ldots, 1$. Therefore $E(T) = \infty$ and T is biased for $1/p$.

Since $b_{1/p}(T) = E(T) - 1/p = \infty$, the MSE of T is also infinite.

iii) Since $1/p$ is the expected time until first failure, an unbiased estimator could be obtained by taking n fuses, installing them in identical appliances and repeatedly switching the appliances on & off until all the fuses have failed for each fuse, count the number of power cycles until failure, & use the mean number of cycles as an estimate of $1/p$.
(OTHER ANSWERS POSSIBLE)